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Flux stabilization in compact groups

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Abstract: We consider the Born-Infeld action for symmetry-preserving, orientable D-branes in compact group manifolds. We find classical solutions that obey the flux quantization condition. They correspond to conformally invariant boundary conditions on the world sheet. We compute the spectrum of quadratic fluctuations and find agreement with the predictions of conformal field theory, up to a missing level-dependent truncation. Our results extend to D-branes with the geometry of twined conjugacy classes; they illustrate the mechanism of flux stabilization of D-branes.

Keywords: D-branes, Conformal Field Models in String Theory.

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1. Introduction

The study of D-branes has deepened our understanding of both string theory and conformal field theory. Indeed, much of the usefulness of D-branes comes from the fact that they can be investigated from two rather different points of view: on the one hand they are described by boundary conditions on open strings. This allows to investigate them using two-dimensional conformal field theories for which many exact methods are available. On the other hand, their worldvolume theories can be studied with conventional field theoretical methods; this gives rise, in particular, to a rich interrelation with Yang-Mills theories.

D-branes in group manifolds are a particularly tractable testing ground; they have recently received much attention. A large class of D-branes in WZW theories that is compatible with the symmetry encoded in the non-abelian currents has a rather simple description: their worldvolumes are (twined) conjugacy classes that obey an integrality condition. This result has first been established by two-dimensional methods [1, 5]. It has been rederived from the Born-Infeld action, for SU(2) [2] and for SU(N) [10].

One surprisingly strong result of [2] was the good agreement of the results obtained from the Born-Infeld action with the exact results from conformal field theory. Indeed, up to a shift of the weights by the Weyl vector and of the level by the dual Coxeter number, the position of the stable solutions of the Born-Infeld action coincide with the exact CFT results. Moreover, the spectrum of quadratic fluctuations is close to the spectrum of boundary fields; the latter, however, is truncated at finite level, an effect that is not reproduced by the Born-Infeld action. There have been speculations that the good agreement of the results in the case of SU(2) is due to the fact that the supersymmetric version of this theory appears in the description of the NS-fivebrane so that effectively a hidden supersymmetry is responsible for the agreement. In the present note, we use the Born-Infeld approach to investigate D-branes on general compact Lie groups and see that the agreement holds in general. This shows in particular that the agreement should not be seen as a miracle of string theory, but rather of conformal field theory. (This can already be seen from the simple fact that the Virasoro central charge of these theories can be arbitrarily high so that most of them cannot appear as building blocks of string theories.)

It is known that compact connected Lie groups are quotients of the product of a semi-simple simply-connected group and a torus by a subgroup of the center. The flux quantization mechanism [2] requires a non-constant B-field; for simplicity we therefore suppress the toroidal part in our discussion; our results, however, obviously extend to an arbitrary real compact Lie group. An important aspect of our considerations is that we only use those general group theoretic methods that can also shed light on more general, non-compact groups.

This note is organized as follows: in section 2 we briefly discuss the Born-Infeld action and derive a useful description of the Kalb-Ramond background field B. In section 3 we impose the flux quantization condition to find candidates for solutions of the equations of motion for the Born-Infeld action. We compute first order and second order fluctuations around these configurations. The first order fluctuations vanish which shows that we have indeed found solutions. The spectrum of the second order fluctuations is positive definite, which shows the stability of the solutions. We compare the spectrum of quadratic fluctuations to the spectrum of operators on the worldvolume of the D-brane that has been obtained in conformal field theory calculations. In section 4 we briefly sketch the generalization of our results to D-branes that are twined conjugacy classes. We finally present our conclusions. Some calculations are relegated to an appendix.

After completion of this work, related results for $\mathrm{G} = \mathrm{SU}(N)$ have appeared in appendix A of [10]. In particular, similar expressions for the background fields B and F have been presented; our results include also expressions for the background fields that are adapted to the investigation of twisted boundary conditions.

2. The Born-Infeld action and the background fields

The Born-Infeld action is a generalization of the minimal surface problem to spaces M that carry not only a metric G but also a two-form gauge field B with three-form field strength $H = dB$. More precisely, one considers embeddings p of a space Σ

with a two-form field strength $F = dA$ into M, with the action

$$S_{BI}(p, F) = \int_{\Sigma} \sqrt{\det(p^*G + p^*B + 2\pi F)}\,. \tag{2.1}$$

The Born-Infeld action involves an integration over the world volume of the brane; we will therefore restrict ourselves to branes that are orientable.[1] The action leads to a variational problem in which both the abelian gauge field A and the embedding p are varied. The fact that the two-form F comes from a field strength implies the following quantization condition:

$$[F] \in H^2(\Sigma, \mathbb{Z})\,, \tag{2.2}$$

i.e. the class of F must be an integral element of the second cohomology. Actually, the field strength F transforms under gauge transformations $B \to B + \Lambda$ of the two-form gauge-field B like $F \to F - \frac{1}{2\pi}p^*\Lambda$. Equivalently, the transformation of its vector potential A is $A \to A + d\theta - \frac{1}{2\pi}p^*\alpha$, with θ a function and α a one-form such that $d\alpha = \Lambda$. However, every regular conjugacy class is contained in a single coordinate patch for the B-field, which allows us to neglect large gauge transformations for our purposes. For a more careful discussion of these issues we refer to [7]. We study the action (2.1) in the case when $M = \mathrm{G}$ is a compact, semi-simple, connected, real Lie group and when Σ is a homogeneous space of the form G/T, where T is a maximal torus of G.

Our first task is to find explicit expressions for the background fields on the group manifold G. To this end we need to fix a non-degenerate, ad-invariant, bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbf{g} = \mathrm{Lie}\,\mathrm{G}$. Such a form is indeed a crucial ingredient to have a conformal field theory with a group manifold as a target space; we therefore pause to present a few comments about it. It has been shown [6] that a general, not necessarily compact, Lie group gives rise to a conformal field theory with all non-abelian currents as symmetries if and only if its Lie algebra admits such a form. (Note, though, that this form can be different from the Killing form; this is necessarily the case for non-semi-simple Lie groups.)

The existence of such a form also allows to reconcile the two seemingly different sets of labels for boundary conditions. From conformal field theory, it is known that conformally invariant boundary conditions correspond quite generally to representations of the chiral symmetry (see [8] for the most general statements). In our case, the D-branes on a group manifold correspond to (twisted) representations of the group. The space-time approach, on the other hand, gives conjugacy classes which, of course, are the orbits of the *adjoint* action of a Lie group G on itself. Representations, in contrast, can be obtained from the quantization of *co-adjoint* orbits (for a

[1]The known D-branes in WZW theories are not all orientable; unorientable D-branes appear in non-simply connected groups. They are quotients of conjugacy classes of the simply connected cover $\tilde{\mathrm{G}}$ that are invariant under the action of a subgroup of the center of $\tilde{\mathrm{G}}$.

review see [9]). Understanding this relation thus requires a natural relation between the adjoint and the co-adjoint action of G on its Lie algebra **g**. This is achieved by a non-degenerate, invariant, bilinear form $\langle\cdot,\cdot\rangle$ on **g**.

The Lie algebra **g** is a direct sum of pairwise orthogonal simple ideals, $\mathbf{g} = \oplus_{i=1}^{n}\mathbf{g}_i$. On each simple ideal, we normalize the Killing form such that for x, y in the same simple ideal $\mathbf{g}_i$ we have

$$\kappa_i(x,y) = -\frac{1}{8\pi g_i^\vee}\,\mathrm{Tr}(ad_x ad_y)\,, \tag{2.3}$$

where $g_i^\vee$ is the dual Coxeter number of $\mathbf{g}_i$. A general invariant bilinear form on **g** is of the form

$$\langle v,w\rangle = \sum_{i=1}^{n} k_i\kappa_i(P_i v, P_i w)\,, \tag{2.4}$$

where P_i is the orthogonal projection to the ideal $\mathbf{g}_i$. As is well-known, consistency of the WZW action requires the numbers k_i to be quantized. For a simply connected group, they have to be integers. For non-simply connected groups, further conditions have to be imposed: e.g. for $\mathrm{G} = \mathrm{SO}(3)$, k has to be even. The n-tuple of non-negative integers (k_i) will be called the level.

Let us now give the background fields in terms of the bilinear form (2.4). We endow the manifold G with the G-bi-invariant metric

$$G(u,v) = \left\langle g^{-1}u, g^{-1}v\right\rangle\,, \tag{2.5}$$

and the three-form field strength

$$H(u,v,w) = -\left\langle g^{-1}u, \left[g^{-1}v, g^{-1}w\right]\right\rangle\,, \tag{2.6}$$

where $u, v, w \in T_g\mathrm{G}$. Due to the normalization of the bilinear form (2.4), the three-form H is in $H^3(\mathrm{G},\mathbb{Z})$.

We wish to find a two-form potential B for H. To this end we use the well-known fact that for every compact connected Lie group the map

$$\begin{aligned} q:\ \frac{\mathrm{G}}{\mathrm{T}}\times\mathrm{T} &\longrightarrow \mathrm{G}\,,\\ q(gT,t):\ &=\ gtg^{-1} \end{aligned} \tag{2.7}$$

is of mapping degree $|W|$, where $|W|$ is the number of elements in the Weyl group W of G. In particular, q is surjective.

Recall that a regular element of G is an element that is contained in only one maximal torus. In this note, we will restrict ourselves to D-branes whose world-volume is contained in the subset G_r of regular elements. (The set regular elements G_r forms an open dense subset of G; the codimension of its complement is at least 3.) The elements of a fixed conjugacy class are either all regular or not; in the

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former case, the conjugacy class will be called regular. Consider $\mathrm{T}_r = \mathrm{G}_r \cap \mathrm{T}$, i.e. the regular elements in the maximal torus; these elements constitute the interior of the Weyl chambers, on which the action of the Weyl group W is free. Each regular conjugacy class intersects the maximal torus in T_r; the different intersections form an orbit of the Weyl group. The map q thus allows us to see G_r topologically as a product of G/T over the interior of one Weyl chamber, i.e. over T_r/W. The notation

$$q_t(gT) := q(gT, t)$$

for a family of maps from G/T to G will come in handy.

To write down B, we introduce two more objects. The first object is a family F_t of two-forms on G/T, parametrized by an element $t \in \mathrm{T}_r$:

$$2\pi F_t(u,v) := 2\left\langle \exp^{-1} t, \left[h^{-1}u, h^{-1}v\right]\right\rangle , \tag{2.8}$$

where $h \in \mathrm{G}$ is an arbitrary representative of the point $hT \in \mathrm{G/T}$ and $u, v \in T_{hT}(\mathrm{G/T})$. For each value of the parameter t the two-form F_t on G/T is closed. The vector $\exp^{-1} t \in \mathbf{g}$ is, of course, only defined up to elements of the lattice Ker(exp). (The ambiguity in (2.8) in choosing a preimage for the exponential map will be discussed at the end of this section.) For an arbitrary compact group this lattice is called the integral lattice. For simply-connected groups it coincides with the co-root lattice, whereas for non-simply-connected groups the integral lattice contains the co-root lattice. The finest possible integral lattice is the one of the adjoint group and equals the co-weight lattice.

The second object we need is a two-form ω on G_r [1]. We define ω_g to be non zero only on vectors $v \in T_g\mathrm{G}$ for which $g^{-1}v$ is in the image of $1 - \mathrm{Ad}_g$, i.e. on the vectors tangent to the conjugacy class that contains g. For such vectors, we set

$$\omega_g(u,v) = \left\langle g^{-1}u, \frac{1+\mathrm{Ad}_g}{1-\mathrm{Ad}_g} g^{-1}v \right\rangle . \tag{2.9}$$

We then give the two-form potential B on G_r by its pull-back to $\mathrm{G/T} \times \mathrm{T}_r$:

$$q^*B_{|(g\mathrm{T},t)} = q^*\omega_{|(g\mathrm{T},t)} - 2\pi (F_t)_{|gT} , \tag{2.10}$$

where the two-form F_t on G/T is extended to a two-form on $\mathrm{G/T} \times \mathrm{T}_r$ by setting it to zero on directions tangent to T_r. One can check that indeed $dB = H$ on G_r (for details, see appendix A).

To conclude this section, we explain why choosing another Lie algebra element for $\exp^{-1} t$ in the definition of F_t (2.8) does not affect the physics of the system. We already noted that the fields B and F are subject to large gauge transformations that have, of course, to be compatible with the quantization of the level which guarantees that $[H] \in H^3(\mathrm{G}, \mathbb{Z})$. One can check that a change of the Lie algebra element

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$\exp^{-1} t$ in (2.8) corresponds to such an allowed gauge transformation, if and only if this change is in the integral lattice. Changing the representatives for $\exp^{-1} t$ thus amounts to performing a large gauge transformation.[2]

3. Flux quantization and classical solutions

To obtain candidates for a solution of the equations of motion of (2.1) whose world-volume is of the form G/T, we use q_t as the embedding and F_t as the field strength. This idea, as well as the form of F_t are inspired by Kirillov's method of coadjoint orbits [9]. To qualify as a field strength, F_t has to be quantized as in (2.2). We first investigate for which t this is the case.

The condition (2.2) amounts to the statement that the integral of the two-form F_t over any two-sphere in the world volume G/T of the D-brane is an integer. A basis for $H_2(\mathrm{G/T}, \mathbb{Z})$ can be obtained as follows: for any simple root α, consider the corresponding su(2) subalgebra $\mathbf{g}_\alpha$ spanned by the co-root H^α and suitable linear combinations of the step operators $E^{\pm\alpha}$. Its image in G under the exponential map is either isomorphic to SU(2) or to SO(3). In both cases, quotienting by T gives a two-sphere S_α in G/T. The spheres S_α generate $H_2(\mathrm{G/T})$. This reduces the problem to the well-known case of SU(2): the integral

$$\int_{S_\alpha} F_t$$

is an integer iff $\exp^{-1} t$ is in the lattice dual to the co-weight lattice; here the duality is with respect to the form (2.4). If we use the standard Killing form to identify the Cartan subalgebra $\mathbf{g}_0$ of $\mathbf{g}$ with its dual $\mathbf{g}_0^*$, the weight space, then the component of $P_i(\exp^{-1} t)$ in the ideal $\mathbf{g}_i$ is of the form $2\pi\lambda/k_i$, where λ is an integral weight of $\mathbf{g}_i$.

To see that we have indeed solutions of the classical equations of motion, we compute the first order and second order fluctuations of the action under variations of the gauge field A on the brane and of the embedding.

We begin with first order fluctuations: with the notation $M_t = q_t^* G + q_t^* B + 2\pi F_t$, we have

$$\delta^{(1)} S_{BI}(q_t, F_t) = \frac{1}{2} \int_{\mathrm{G/T}} \sqrt{\det(M_t)}\, \mathrm{Tr}\left(M_t^{-1} \delta M_t\right). \tag{3.1}$$

For a first order variation δA of the one-form gauge field, one has

$$\begin{aligned} \delta_F^{(1)} S_{BI}(q_t, F_t) &= \int_{\mathrm{G/T}} \sqrt{\det(M_t)}\, \mathrm{Tr}\left(M_t^{-1} d(\delta A)\right) \\ &= -\int_{\mathrm{G/T}} \mathrm{Tr}\left(\delta A\ d\left(\sqrt{\det(M_t)} M_t^{-1}\right)\right). \end{aligned}$$

[2]Note that our gauge choice respects the quantization of $[H]$, in the sense defined in [2]; our B-field vanishes at the origin.

An explicit calculation (cf. appendix A) shows that

$$d\left(\sqrt{\det(M_t)}M_t^{-1}\right) = 0\,,$$

so that first order fluctuations of the gauge field around our configurations vanish. As for the fluctuations of the embedding p, one can show that for any fluctuation $\delta_p M_t$ one has the identity

$$\mathrm{Tr}\left(M_t^{-1}\delta_p M_t\right) = 0\,,$$

which implies that $\delta_p^{(1)} S_{BI} = 0$. This shows that we have indeed found classical solutions. Their positions form a lattice which we write in terms of the weight lattice L_i^w of each simple ideal $\mathbf{g}_i$, up to the above-mentioned identification of $\mathbf{g}_0$ with $\mathbf{g}_0^*$:

$$L = \oplus_{i=1}^n \frac{2\pi}{k_i} L_i^w\,. \tag{3.2}$$

Recall that the set of conjugacy classes is parametrized by T_r/W, or, via the exponential map, by $\mathbf{g}_0/\hat{W}$, where $\hat{W}$ is the affine Weyl group, $\hat{W} = W \ltimes \mathrm{Ker}(\exp)$. Therefore, elements of L that are related by the action of the affine Weyl group are in fact identical solutions. Inequivalent solutions are thus parametrized by the coset $L/\hat{W}$.

Our result for the positions coincides, up to a shift of the weights by the Weyl vector and a shift of the level by the dual Coxeter number, with the exact results [5] of conformal field theory.

To compute the quadratic fluctuations, we introduce coordinates x^i on G/T and a lift h from G/T to G, i.e. $h = h(x^i) \in \mathrm{G}$, and coordinates ψ^α on T, i.e. $t(\psi^\alpha) \in \mathrm{T}$. For any choice of $t \in \mathrm{T}$, we then define e_i as the orthogonal projection of

$$m_i(x) := (\partial_{x^i} h)h^{-1} \tag{3.3}$$

on $\mathrm{Im}(1 - \mathrm{Ad}_{hth^{-1}})$. Neither the vector m_i nor the image depend on t, since t is a regular element of the maximal torus. We now use the invariant metric (2.4) to define the metric tensor

$$\gamma_{ij}(x) = \langle e_i(x), e_j(x)\rangle \tag{3.4}$$

on the tangent space of $T_{h(x)\mathrm{T}}(G/T)$. Moreover, considering $g = ht(\psi^\alpha)h^{-1}$ as a function of ψ^α gives us vectors

$$u_\alpha := g^{-1}\partial_{\psi^\alpha} g\,,$$

that are a basis of $\mathrm{Ker}(1 - \mathrm{Ad}_g)$, the orthogonal complement of $\mathrm{Im}(1 - \mathrm{Ad}_g)$ in $\mathbf{g}$. The basis (e_i) of $\mathrm{Im}(1 - \mathrm{Ad}_g)$ allows us to consider any two-tensor A_{ij} on G/T as a linear operator A on $\mathrm{Im}(1 - \mathrm{Ad}_g)$, such that

$$A_{ij} = \langle e_i, A(e_j)\rangle\,;$$

then its trace is $\mathrm{Tr}(A) = \langle e^i, A(e_i)\rangle$ (indices are raised by γ_{ij}) and the inverse tensor corresponds to the inverse operator.

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Considering the ψ^α as real functions on G/T describes a general embedding of G/T in the neighborhood of a conjugacy class. We can thus parametrize an infinitesimal variation of the embedding p around the fixed embedding q_t by functions $\delta\psi^\alpha$. The fluctuations of the gauge field are described in terms of fluctuations δA of the gauge potential A. Note that the functions $\delta\psi^\alpha$ are scalar functions on G/T while the function δA is a one-form on G/T.

To write down the equations of motion, we introduce a covariant derivative ∇ on G/T which in the coordinates x^i reads

$$\nabla^i = \frac{1}{\sqrt{\det\gamma}}\partial_j\sqrt{\det\gamma}\gamma^{ij}\,. \tag{3.5}$$

The group G acts on the space of functions on G/T by left translation and thus provides a quadratic Casimir operator $\Box$ which coincides with the laplacian of the connection ∇^i:

$$\Box = \nabla^i\partial_i\,.$$

We are now ready to give the second order fluctuations,

$$\begin{aligned}\delta^{(2)}S_{BI}(t) = \int_{\mathrm{G/T}} & dx^i\sqrt{\det\left(M_t\right)}\times\\ \times & \left[\frac{1}{8}\gamma^{ij}\partial_i\delta\psi^\alpha\partial_j\delta\psi^\beta\delta_{\alpha\beta} - \frac{1}{8}\operatorname{Tr}\left(ad_{u_\alpha}ad_{u_\beta}\right)\delta\psi^\alpha\delta\psi^\beta - \right.\\ & -\frac{1}{4}\operatorname{Tr}\left(\delta F\frac{2}{1-\mathrm{Ad}_g}\delta F\frac{2}{1-\mathrm{Ad}_g}\right) - \\ & \left. -\frac{1}{8}\operatorname{Tr}\left(\delta F ad_{u_\alpha}\right)\delta\psi^\alpha + \frac{1}{8}\operatorname{Tr}^2\left(\frac{2}{1-\mathrm{Ad}_g}\delta F\right)\right].\end{aligned} \tag{3.6}$$

This implies the equations of motion:

$$\begin{aligned}\delta\psi^\alpha:\ & \delta_{\alpha\beta}\Box\delta\psi^\alpha + \operatorname{Tr}\left(ad_{u_\alpha}ad_{u_\beta}\right)\delta\psi^\alpha + \frac{1}{2}\operatorname{Tr}\left(\delta F ad_{u_\beta}\right) = 0\,,\\ \delta A_j:\ & -4(ad_{u_\alpha})^{ij}\partial_j\delta\psi^\alpha + \frac{1}{2}\left\langle e^i,\left[e^k,e^l\right]\right\rangle\delta F_{kl} + \nabla^k\delta F_k{}^i = 0\,.\end{aligned} \tag{3.7}$$

These equations should be complemented by the gauge condition

$$\nabla(\delta A) = 0\,, \tag{3.8}$$

which is invariant under the action of G. It should be appreciated that neither the equations of motion nor the gauge condition depend on the position t of the brane. This is in accordance with the result from conformal field theory [5] that in the large level limit the fields on a brane do not depend on its position.

Using the fact that e_i, u^α form a basis of $\mathbf{g}$, we parametrize the fluctuations δA and $\delta\psi^\alpha$ in terms of a single function $f:\mathrm{G/T}\to\mathbf{g}$:

$$\delta A_i = 2\left\langle e_i, f\right\rangle\,,\qquad \delta\psi^\alpha = \left\langle u^\alpha, f\right\rangle\,, \tag{3.9}$$

The gauge condition (3.8) then translates into

$$\left\langle e_i, \partial^i f \right\rangle = 0 \tag{3.10}$$

while the equations of motion (3.7) become

$$\begin{aligned}\langle u_\beta, \Box f \rangle &= 0\,,\\ \langle e_i, \Box f \rangle &= 0\end{aligned} \tag{3.11}$$

which amounts to

$$\Box f = 0\,. \tag{3.12}$$

We now consider how the solutions of (3.12) and (3.10) transform under the left action of G. The function f itself transforms in the adjoint representation R_θ. Since the differential operator in (3.12) is a singlet under the action of G, the solutions of (3.12) come in representations of the form $R_\theta \otimes R_0$ where R_0 is any representation of $\mathbf{g}$ with vanishing second order Casimir eigenvalue. The gauge condition (3.10), on the other hand, is a $\mathbf{g}$-covariant map from $R_\theta \otimes R_0$ to R_0; thus it removes one representation isomorphic to R_0 from the solutions $R_\theta \otimes R_0$.

This result is, in itself, of limited interest as long as we take G to be a compact group: in this case, the only representation of Casimir-eigenvalue 0 is the trivial, one-dimensional, representation. However, all our computations continue to hold if we take the group to be the product $\mathrm{G}_{\mathrm{tot}} = \mathrm{G} \times \mathbb{R}$ of a compact group G with a time-like factor $\mathbb{R}$ with negative metric. (One could also take a product with some other non-compact group, see [3, 11].) The D-brane is supposed to extend along the time-like direction, while its spatial position is still parametrized by the regular elements of the maximal torus of the compact group G. The contribution of $\mathbb{R}$ to the total Casimir eigenvalue of $\mathrm{G}_{\mathrm{tot}} = \mathrm{G} \times \mathbb{R}$ takes arbitrary negative values that can cancel any positive contribution from G. In this case, all unitary representations R of G appear in the spectrum.

The fluctuation spectrum including a timelike direction can be compared with results from conformal field theory, as in [2]. We read off that the multiplicity of $R_\theta \otimes R - R$ in the spectrum of quadratic fluctuations equals the multiplicity of R in the space of functions on $\mathrm{G}_{\mathrm{tot}}/\mathrm{T}$, which in turn is equal to the multiplicity of the corresponding representation in the space of functions on G/T. This result differs, as in the case of SU(2), from the exact CFT result [5]: for the latter, a level-dependent truncation of the representations R is found. Only representations that are integrable at the given level appear in the exact result. The energy eigenvalues of the quadratic fluctuations are of the form C_R/k, where C_R is the eigenvalue of the second order Casimir operator on the representation R. They are positive, which shows the stability of our solutions; moreover, they agree, as in the case of SU(2), with the conformal weights in the limit of large k.

To discuss the energies of the branes themselves, let us restrict for simplicity to a single simple factor. The energy E_λ of a D-brane at $2\pi\lambda/k$ is proportional to the quantum dimension of the corresponding boundary condition:

$$D_\lambda = \prod_{\alpha>0} \frac{\sin\left(\frac{\pi(\lambda+\rho,\alpha)}{k+g^\vee}\right)}{\sin\left(\frac{\pi(\rho,\alpha)}{k+g^\vee}\right)}\,, \tag{3.13}$$

where the product is over the positive roots α, and the parenthesis denote the standard bilinear form on the Cartan subalgebra of $\mathbf{g}_i$, normalized such that the highest root has length squared 2. Indeed, the direct calculation of E_λ with $\exp^{-1} t = 2\pi\lambda/k$ gives

$$E_\lambda = S_{BI}(q_t, F_t) \propto \int_{\mathrm{G/T}} \sqrt{\det \gamma_{ij}}\,\sqrt{\det(1-\mathrm{Ad}_t)}\,, \tag{3.14}$$

where the operator $(1-\mathrm{Ad}_t)$ has to be restricted to its image. We find [3]

$$E_\lambda \propto \prod_{\alpha>0} \sin\left(\frac{\pi}{k}(\lambda,\alpha)\right). \tag{3.15}$$

The results of this section can be summarized by the statement that, in the limit of large level, the Born-Infeld results agree with the exact CFT results for all compact Lie groups.

4. Generalization to twisted boundary conditions

Our considerations can be easily generalized to D-branes that are twined conjugacy classes [5] $\mathcal{C}^\omega(h) = \{gh\omega^{-1}(g)\}$ where ω is an automorphism of G.[4] This automorphism relates left movers and right movers at the boundary and determines the automorphism type of the boundary conditions. Twined conjugacy classes can be parametrized as follows [5]: any automorphism ω leaves at least one maximal torus T of G invariant. Denote by T_0^ω the connected component of the identity of the subgroup of T that is left pointwise invariant by ω: for compact G, this is a proper subgroup in the case of an outer automorphism. Up to the action of the subgroup of the Weyl group consisting of elements that commute with ω, T_0^ω parametrizes the twined conjugacy classes. Regular twined conjugacy classes are diffeomorphic to $\mathrm{G/T}_0^\omega$; in particular, for ω an outer automorphism their dimension is greater than the dimension of regular conjugacy classes.

[3] The λ-independent prefactors were computed in [10] for $\mathrm{G}=\mathrm{SU}(N)$ and precise agreement was found in the limit of large level.

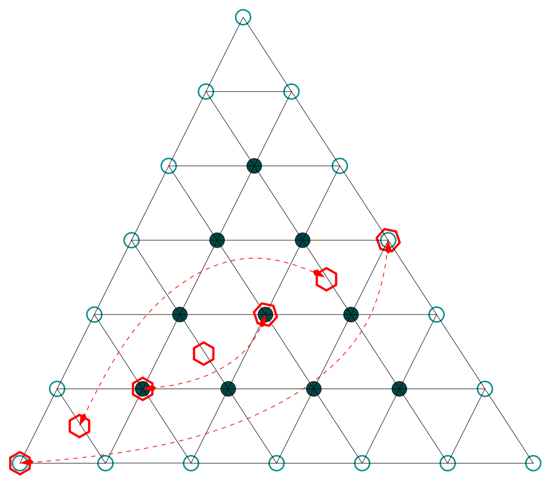
[4] We thank the referee for drawing our attention to the fact that shortly after publication of [5] twined conjugacy classes have also been discussed in [12].

The expressions for the background fields can be generalized as follows (see also [13]):

$$\omega_g(u,v) = \left\langle g^{-1}u, \frac{1+\omega\circ \mathrm{Ad}_g}{1-\omega\circ \mathrm{Ad}_g} g^{-1}v \right\rangle, \tag{4.1}$$

where now $g^{-1}u \in \mathrm{Im}(1-\omega\circ\mathrm{Ad}_g)$. The family F_t^ω is now a family of two-forms on $\mathrm{G}/\mathrm{T}_0^\omega$ parametrized by T_0^ω. Formally, we take the same expression as in (2.8), with the important difference that now t is required to be in T_0^ω.

A similar argument as for the untwisted case shows that the quantization condition (2.2) implies that $\exp^{-1} t$ is a fractional symmetric weight, which agrees with the results in [4, 5]. (Note that there is an additional shift of the weight lattice for $\mathrm{SU}(N)$ with N odd which comes from extending the automorphism ω from $\mathrm{Lie}\,\mathrm{G}$ to the untwisted affine Lie algebra. It is not seen in the Born-Infeld action, which does not have a direct relation to the affine Lie algebra.)



(4.2)

We illustrate our results in the case of SU(3) at level $k = 6$: figure 4.2 shows the intersections of the brane worldvolumes with the fundamental Weyl chamber in $\mathrm{Lie}\,\mathrm{T}$. Light circular vertices mark the intersections of singular branes with the boundary of the Weyl chamber. Full dark circles denote regular six-dimensional conjugacy classes on which a D-brane can wrap. Hexagonal marks give the intersection of the seven-dimensional worldvolume of twined conjugacy classes with the fundamental Weyl chamber; they are all contained in the one-dimensional subspace $\mathrm{Lie}\,\mathrm{T}_0^\omega$. One and the same *twined* brane can intersect the fundamental Weyl chamber in more than one point; the corresponding hexagonal marks are linked by arrows.

The calculation of the spectrum of quadratic fluctuations can be generalized as well. To this end, it is necessary to realize that $\mathrm{Lie}\mathrm{T}_0^\omega = \mathrm{Ker}(1-\omega\circ\mathrm{Ad}_t)$. We then use the operator $1-\omega\circ\mathrm{Ad}_t$ instead of $1-\mathrm{Ad}_t$ to define e_is and u_αs, as above. In every subsequent expression with a dependence in Ad_g, the latter should be replaced by $\omega\circ\mathrm{Ad}_g$. In particular, we obtain a metric $\gamma_{ij} = \langle e_i, e_j\rangle$ on $\mathrm{G}/\mathrm{T}_0^\omega$ whose laplacian coincides with the quadratic Casimir of the action of G on the space of functions on G/T_0^ω by left translation. The resulting spectrum is expressed in terms of eigenvectors of this operator to the eigenvalue zero. The action of ω does not change the formal structure of the result, but, of course, modifies the multiplicities of the irreducible representations: we now have to decompose the space of functions on G/T_0^ω (the multiplicities are given in [5, (4.21)]). The resulting multiplicities still match the corresponding annulus amplitude, up to the missing usual level-dependent truncation.

5. Conclusions

In this note, we have studied solutions of Born-Infeld theory on compact, connected Lie groups G that have the form of (twined) conjugacy classes. We have found explicit expressions for the two-form gauge field B on regular group elements and the two-form field strength F on regular conjugacy classes. These expressions may be generalized to other cases, including non-compact groups, provided their Lie algebra admits an invariant non-degenerate bilinear form. However, in the case of non-compact groups, even if they are semi-simple, one has to be careful with the definitions of regular elements and Cartan subgroups.

The quantization condition on F implies a quantization on the possible positions of the branes; they agree, up to a shift, with the exact CFT results. The energies of the D-branes are proportional to the quantum dimensions of the corresponding boundary conditions.

Our results give us confidence that also on more general Lie groups (and manifolds with possibly even less structure), solutions of the Born-Infeld action can provide information on boundary conditions of conformal field theories.

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A. Some explicit calculations

From the main text, we recall the notations $h(x^i) \in G$ for the local lift from G/T to G, $t(\psi^\alpha) \in T$, and finally $g = hth^{-1} \in G$. We use latin indices for directions tangent to G/T; they are raised and lowered with the metric γ_{ij} (3.4); greek indices stand for directions tangent to the maximal torus. We denote $C_g = C_t$ the conjugacy class of g and t, respectively.

Calculation of $dB = H$: The two-form B on G_r is defined by (2.10); we rewrite F_t as

$$2\pi F_t = \left\langle exp^{-1}t, \left[h^{-1}\partial_i h, h^{-1}\partial_j h\right]\right\rangle dx^i \wedge dx^j \,.$$

This two-form on G/T is closed, since $d([h^{-1}dh, h^{-1}dh]) = -d^2(h^{-1}dh) = 0$. Next, we rewrite:

$$\omega_g = \left\langle g^{-1}\partial_i g, \frac{1+Ad_g}{1-Ad_g} g^{-1}\partial_j g\right\rangle dx^i \wedge dx^j \,. \tag{A.1}$$

It is known [1] that $d\omega_{|C_g} = H_{|C_g}$; we thus only need to calculate the components of the differential of $(q_t^*\omega - 2\pi F_t)$ tangent to the maximal torus:

$$\begin{aligned} \partial_\alpha 2\pi (F_t)_{ij} &= 2\left\langle t^{-1}\partial_\alpha t, \left[h^{-1}\partial_i h, h^{-1}\partial_j h\right]\right\rangle, \\ \partial_\alpha \omega_{ij} &= -\left\langle \left[t^{-1}\partial_\alpha t, \mathrm{Ad}_g\left(h^{-1}\partial_{[i} h\right)\right], h^{-1}\partial_{j]} h\right\rangle. \end{aligned} \tag{A.2}$$

They are to be compared to the expression for $H_{\alpha ij}$:

$$\begin{aligned} H_{\alpha ij} &= 2\left\langle t^{-1}\partial_\alpha t, \left[h^{-1}\partial_i h, h^{-1}\partial_j h\right]\right\rangle - \\ &\quad -\left\langle t^{-1}\partial_\alpha t, \left[\mathrm{Ad}_g\left(h^{-1}\partial_i h\right), h^{-1}\partial_j h\right] + \left[h^{-1}\partial_i h, \mathrm{Ad}_g\left(h^{-1}\partial_j h\right)\right]\right\rangle. \end{aligned}$$

The second term coincides with $\partial_\alpha \omega_{ij}$, while the first one equals indeed $2\pi\partial_\alpha (F_t)_{ij}$. Finally, we notice that, for any index M, we have $H_{\alpha\beta M} = 0$. Since $B_{\alpha M} = 0$, we have proved $dB = H$.

Calculation of $\mathrm{Tr}(M_t^{-1}\delta_\alpha M_t) = 0$: We use the projection e_i of $\partial_i h\ h^{-1}$ on $\mathrm{Im}(1 - \mathrm{Ad}_g)$ to write $(M_t)_{ij} = 2\langle (1-\mathrm{Ad}_g)e_i, e_j\rangle$. The geometrical fluctuation of M_t is

$$\delta_\alpha (M_t)_{ij} = -2\left\langle \left[g^{-1}\partial_\alpha g, (\mathrm{Ad}_g - 1)e_i\right], e_j\right\rangle.$$

Therefore, using the identity $(\mathrm{Ad}_g - 1)g^{-1}\partial_\alpha g = 0$ in the last equality, we obtain:

$$\begin{aligned} \mathrm{Tr}\left(M_t^{-1}\delta_\alpha M_t\right) &= \left\langle \frac{1}{1-\mathrm{Ad}_g} e^j, e^i\right\rangle \left\langle (\mathrm{Ad}_g - 1)e_i, \left[g^{-1}\partial_\alpha g, e_j\right]\right\rangle \\ &= \left\langle \frac{\mathrm{Ad}_g - 1}{2\mathrm{Ad}_g} g^{-1}\partial^j g, \left[g^{-1}\partial_\alpha g, \frac{1}{\mathrm{Ad}_g - 1} g^{-1}\partial_j g\right]\right\rangle = 0\,. \end{aligned}$$

Calculation of $d(\sqrt{\det M_t} M_t^{-1}) = 0$: We use the notation $T := M_t^{-1}$ to rewrite:

$$d\left(\sqrt{\det M_t} M_t^{-1}\right) = \sqrt{\det{(M_t)}}\partial_j (M_t)_{rs}\left(T^{[s,r]}T^{[i,j]} - T^{[j,r]}T^{[s,i]}\right)\ dx_i\,. \tag{A.3}$$

In the sequel we will use $T^{ij} = \frac{1}{2}\langle g^{-1}\partial^i g, (1-\mathrm{Ad}_g)g^{-1}\partial^j g\rangle$, where the indices are now raised by the metric $\langle g^{-1}\partial^i g, g^{-1}\partial^j g\rangle$. We also introduce the notation $g^{-1}\partial_r\partial_j g = v_{rj}$.

We can write

$$\begin{aligned}\partial_j (M_t)_{rs} = &\frac{1}{2}\left\langle g^{-1}\partial_j g, \left[g^{-1}\partial_r g, \frac{1}{1-\mathrm{Ad}_g} g^{-1}\partial_s g\right]\right\rangle + \\ &+ \left\langle g^{-1}\partial_r g, \frac{\mathrm{Ad}_g}{\mathrm{Ad}_g - 1}\left[g^{-1}\partial_j g, \frac{1}{1-\mathrm{Ad}_g} g^{-1}\partial_s g\right]\right\rangle + \\ &+ \frac{1}{2}\left\langle g^{-1}\partial_r g, \frac{1}{\mathrm{Ad}_g - 1}\left[g^{-1}\partial_j g, g^{-1}\partial_s g\right]\right\rangle + \\ &+ \left\langle v_{rj}, \frac{1}{\mathrm{Ad}_g - 1} g^{-1}\partial_s g\right\rangle + \left\langle g^{-1}\partial_r g, \frac{1}{\mathrm{Ad}_g - 1} v_{sj}\right\rangle . \end{aligned} \tag{A.4}$$

The contribution of the last line of (A.4) to (A.3) vanishes by itself: using the fact that $v_{rj} = v_{jr}$, we insert the first term of this line in (A.3) and write the result

$$\begin{aligned}&\left\langle v_{rj}, \frac{1}{\mathrm{Ad} - g1} g^{-1}\partial_s g\right\rangle \left\langle g^{-1}\partial^s g, (1-\mathrm{Ad}_g) g^{-1}\partial^r g\right\rangle \left\langle g^{-1}\partial^j g, (1-\mathrm{Ad}_g) g^{-1}\partial^i g\right\rangle = \\ &= -\left\langle v_{rj}, g^{-1}\partial^r g\right\rangle \left\langle g^{-1}\partial^j g, (1-\mathrm{Ad}_g) g^{-1}\partial^i g\right\rangle .\end{aligned}$$

The second term of the same line gives a similar contribution, but with opposite sign.

The contribution of the first three lines of (A.4) to (A.3) also vanishes. To see this, note that all the resulting terms are of the form

$$\left\langle K(\mathrm{Ad}_g) g^{-1}\partial_i g, \left[L(\mathrm{Ad}_g) g^{-1}\partial^i g, M(\mathrm{Ad}_g) g^{-1}\partial^j g\right]\right\rangle ,$$

where K, L, M are some operators on $\mathbf{g}$ built of Ad_g. Then $M(\mathrm{Ad}_g) g^{-1}\partial^j g \in \mathrm{Im}(1 - \mathrm{Ad}_g)$ whereas $[K(\mathrm{Ad}_g) g^{-1}\partial_i g, L(\mathrm{Ad}_g) g^{-1}\partial^i g] \in \mathrm{Ker}(1 - \mathrm{Ad}_g)$, and therefore all these terms vanish.

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